Karnaugh Maps and the Quine-McCluskey Method

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July 15, 2024

Karnaugh Maps

In a Karnaugh map the boolean variables are transferred and ordered in such a way that product terms are easily detected.

		BC			
		00	01	11	10
A	0	m0	m1	m3	m2
	1	m4	m5	m7	m6

4-Variable Karnaugh Maps

AB \ CD	00	01	11	10
00	m0	m1	m3	m2
01	m4	m5	m7	m6
11	m12	m13	m15	m14
10	m8	m9	m11	m10

Grouping minterms

- Minterm rectangles (implicants) should be as large as possible without containing any 0s
- Each side of the rectangle must be a power 2
- The grid is toroidally connected, which means that rectangular groups can wrap across the edges

Example

$$F = AC + A\overline{B}$$

		AB 00	01	11	10
C	0	0	0	0	1
	1	0	0	1	1

Example

$$F = A\overline{B} + \overline{A}CD + AB\overline{C}$$

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	1	1	0	1
10	0	0	0	1

Example

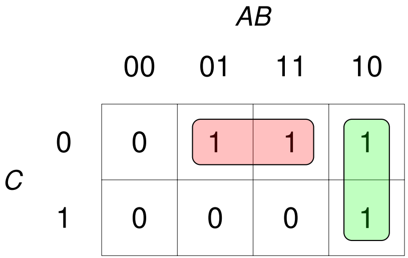
$$F = C + A\overline{D} + \overline{A}B\overline{C}D + A\overline{B}C$$

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	1	0	1
11	1	1	1	1
10	1	1	1	1

Example

Consider the following truth table

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

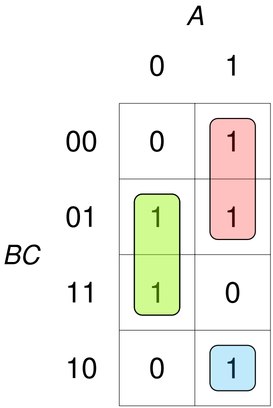


$$F = B\overline{C} + A\overline{B}$$

Example

Consider the following truth table

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0



$$F = \overline{A}C + A\overline{B} + AB\overline{C}$$

Example

Consider the following karnaugh map

CD \ AB	00	01	11	10
00	0	0	1	0
01	0	0	1	1
11	0	0	0	1
10	1	1	0	0

Example

CD \ AB	00	01	11	10
00	0	0	1	0
01	0	0	1	1
11	0	0	0	1
10	1	1	0	0

$$F = AB\overline{C} + \overline{A}C\overline{D} + A\overline{B}D$$

Example

Consider the following karnaugh map

CD \ AB	00	01	11	10
00	1	0	0	0
01	1	0	1	1
11	1	0	1	1
10	1	0	0	0

Example

CD \ AB	00	01	11	10
00	1	0	0	0
01	1	0	1	1
11	1	0	1	1
10	1	0	0	0

$$F = \overline{A}\overline{B} + AD$$

Don't Care Conditions

- In practice there are combinations that will never occur
- we may pick the most convenient assignment

Example

A BCD (Binary Coded Decimal) number is greater than 5

Num	*A*	*B*	*C*	*D*	*F*
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	-
11	1	0	1	1	-
12	1	1	0	0	-
13	1	1	0	1	-
14	1	1	1	0	-
15	1	1	1	1	-

Don't Care Conditions

- In practice there are combinations that will never occur
- we may pick the most convenient assignment

Example

A BCD (Binary Coded Decimal) number is greater than 5

Num	A	B	C	D	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	-
11	1	0	1	1	-
12	1	1	0	0	-
13	1	1	0	1	-
14	1	1	1	0	-
15	1	1	1	1	-

$$F = \Sigma m(6,7,8,9) + d(10,11,12,13,14,15)$$

don't cares = 0

$$F = \overline{A}BC + A\overline{B}\overline{C}$$

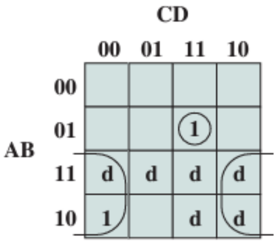
don't cares = 1

$$F = BC + A$$

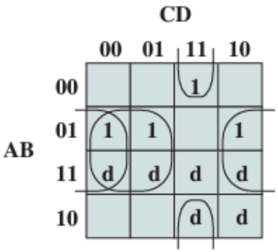
Kmaps with Don't Cares

	Input					Output			
Number	A	B	C	D	Number	W	X	Y	Z
0	0	0	0	0	1	0	0	0	1
1	0	0	0	1	2	0	0	1	0
2	0	0	1	0	3	0	0	1	1
3	0	0	1	1	4	0	1	0	0
4	0	1	0	0	5	0	1	0	1
5	0	1	0	1	6	0	1	1	0
6	0	1	1	0	7	0	1	1	1
7	0	1	1	1	8	1	0	0	0
8	1	0	0	0	9	1	0	0	1
9	1	0	0	1	0	0	0	0	0
Don't care condition	1	0	1	0		d	d	d	d
	1	0	1	1		d	d	d	d
	1	1	0	0		d	d	d	d
	1	1	0	1		d	d	d	d
	1	1	1	0		d	d	d	d
	1	1	1	1		d	d	d	D

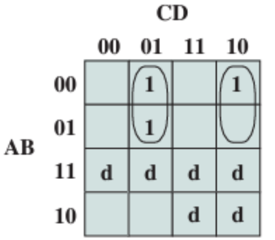
Kmaps with Dont Cares



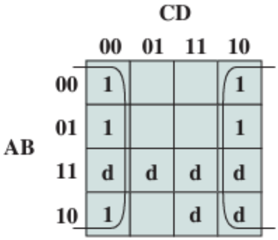
(a) $W = A\overline{D} + \overline{A}BCD$



(b) $X = B\overline{D} + B\overline{C} + BCD$



(c) $Y = \overline{A}\overline{C}D + \overline{A}C\overline{D}$



(d) $Z = \overline{D}$

Shannon Expansion (Decomposition)

Example

Consider the following truth table

$$F = \overline{A}F_1 + AF_2$$

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$F_1 = \overline{C} + B$$

B \ C	0	1
0	1	0
1	1	1

$$F_2 = \overline{B}\,\overline{C} + BC$$

B \ C	0	1
0	1	0
1	0	1

Examples

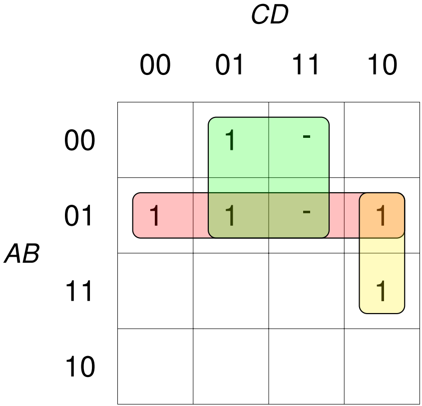
- $F(A, B, C) = \Sigma m(1, 4, 5) + d(0)$
- $F(A, B, C) = \Sigma m(0, 3, 5, 6)$
- $F(A, B, C) = \Sigma m(0, 2, 3, 4, 5, 7)$

Examples

- $F(A, B, C, D) = \Sigma m(1, 4, 5, 6, 14) + d(3, 7)$
- $F(A, B, C, D) = \Sigma m(0, 4, 5, 6, 9, 12, 13)$
- $F(A, B, C, D) = \Sigma m(0, 2, 5, 6, 7, 8, 10, 13)$
- $F(A, B, C, D) = \Sigma m(1, 5, 6, 7, 11, 12, 13, 15)$

Example

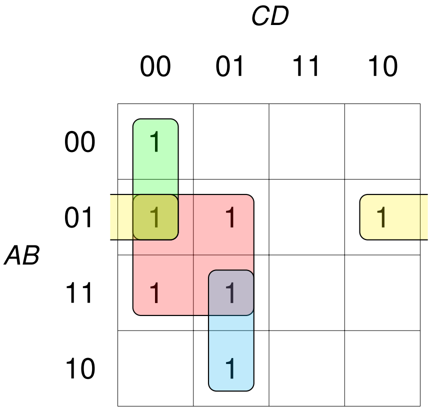
$$F(A,B,C,D) = \Sigma m(1,4,5,6,14) + d(3,7)$$



$$F(A,B,C,D) = \overline{A}B + \overline{A}D + BC\overline{D}$$

Example

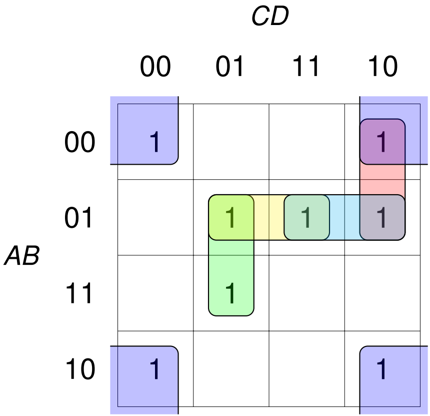
$$F(A,B,C,D) = \Sigma m(0,4,5,6,9,12,13)$$



$$F(A,B,C,D) = B\overline{C} + \overline{A}\overline{C}\overline{D} + A\overline{C}D + \overline{A}B\overline{C}$$

Example

$$F(A,B,C,D) = \Sigma m(0,2,5,6,7,8,10,13)$$



$$F(A,B,C,D) = \overline{B}\,\overline{D} + B\overline{C}D + \overline{A}BC$$

Definitions

Definition (Implicant)

An **implicant** is a ”covering” (product term) of one or more minterms in a sum of products of a boolean function.

Definition (Prime Implicant)

A **prime implicant** of a function is an implicant that cannot be covered by a more general implicant (i.e. an implicant with fewer literals).

Definition (Essential Prime Implicant)

An **essential prime implicant** covers at least one minterm that is not covered by any other prime implicant.

Minimal Two-Level Sum

Definition (Minimal SOP)

An SOP is minimal iff there exist no other SOP with fewer terms.

Theorem

A minimal SOP consists of prime implicants only.

Theorem

Any minimal SOP contains all essential prime implicants.

Algorithm: Minimal Two-Level Sum

1. find all prime implicants
2. find all essential prime implicants
3. find a minimal cover

Remark

The minimal solution may not be unique.

A 5-variable K-map

$F(A, B, C, D) =$
$\Sigma m(1, 4, 5, 6, 9, 10, 13, 17, 20, 21, 22, 25, 29) + \Sigma d(23, 31)$

$A = 0$

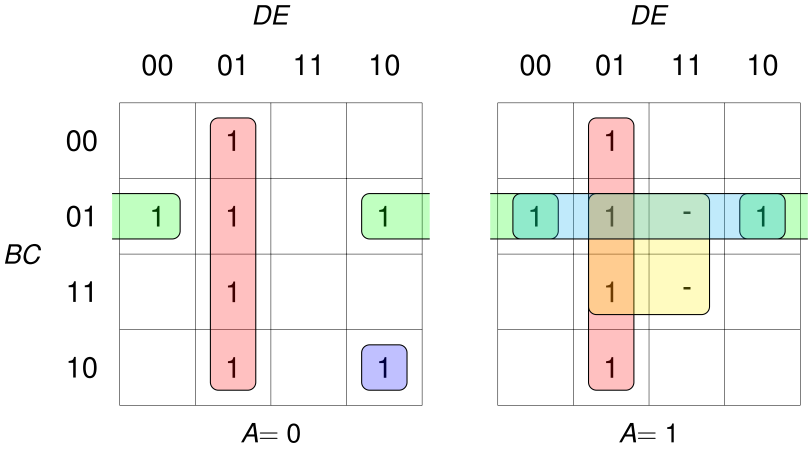
BC \ DE	00	01	11	10
00		1		
01	1	1		1
11		1		
10		1		1

$A = 1$

BC \ DE	00	01	11	10
00		1		
01	1	1	-	1
11		1	-	
10		1		

A 5-variable K-map

$F(A, B, C, D) =$
$\Sigma m(1, 4, 5, 6, 9, 10, 13, 17, 20, 21, 22, 25, 29) + \Sigma d(23, 31)$



A 5-variable K-map

$F(A, B, C, D) =$
$\Sigma m(1, 4, 5, 6, 9, 10, 13, 17, 20, 21, 22, 25, 29) + \Sigma d(23, 31)$

- Prime Implicants: $\overline{D}E$, $\overline{B}C\overline{E}$, $A\overline{B}C$, ACE, $\overline{A}B\overline{C}D\overline{E}$
- Essential Prime Implicants: $\overline{D}E$, $\overline{B}C\overline{E}$, $\overline{A}B\overline{C}D\overline{E}$
- Minmal SOP $F(A, B, C, D) = \overline{D}E + \overline{B}C\overline{E} + \overline{A}B\overline{C}D\overline{E}$

Quine-McCluskey

Example

$F(A, B, C, D) = \Sigma m(0, 1, 3, 7, 8, 9, 11, 15)$

Involves several steps

First get the binary equivalent for the terms

	A	B	C	D
0	0	0	0	0
1	0	0	0	1
3	0	0	1	1
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
11	1	0	1	1
15	1	1	1	1

Quine-McCluskey: First Step

Group the terms

	A	B	C	D
0	0	0	0	0
1	0	0	0	1
3	0	0	1	1
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
11	1	0	1	1
15	1	1	1	1

Group	Minterm	A	B	C	D
0	m_0	0	0	0	0
1	m_1	0	0	0	1
	m_8	1	0	0	0
2	m_3	0	0	1	1
	m_9	1	0	0	1
3	m_7	0	1	1	1
	m_{11}	1	0	1	1
4	m_{15}	1	1	1	1

Quine-McCluskey: Second Step

Match the terms

Group	Minterm	A	B	C	D
0	m_0	0	0	0	0
1	m_1	0	0	0	1
	m_8	1	0	0	0
2	m_3	0	0	1	1
	m_9	1	0	0	1
3	m_7	0	1	1	1
	m_{11}	1	0	1	1
4	m_{15}	1	1	1	1

Group	Match	A	B	C	D
0	m_0 - m_1	0	0	0	_
	m_0 - m_8	_	0	0	0
1	m_1 - m_3	0	0	_	1
	m_1 - m_9	_	0	0	1
	m_8 - m_9	1	0	0	_
2	m_3 - m_7	0	_	1	1
	m_3 - m_{11}	_	0	1	1
	m_9 - m_{11}	1	0	_	1
3	m_7 - m_{15}	_	1	1	1
	m_{11} - m_{15}	1	_	1	1

Quine-McCluskey: Third Step

Match the terms

Group	Match	A	B	C	D
0	m_0 - m_1 m_8 _ m_9	_	0	0	_
	m_0 - m_8 m_1 _ m_9	_	0	0	_
1	m_1 - m_3 m_9 _ m_{11}	_	0	_	1
	m_1 - m_9 m_5 _ m_{11}	_	0	_	1
2	m_3 - m_7 m_{11} _ m_{15}	_	_	1	1
	m_3 - m_{11} m_7 _ m_{15}	_	_	1	1

Quine-McCluskey: Final Step

Group	Match	A	B	C	D
0	m_0 - m_1 m_8 _ m_9	_	0	0	_
	m_0 - m_8 m_1 _ m_9	_	0	0	_
1	m_1 - m_3 m_9 _ m_{11}	_	0	_	1
	m_1 - m_9 m_5 _ m_{11}	_	0	_	1
2	m_3 - m_7 m_{11} _ m_{15}	_	_	1	1
	m_3 - m_{11} m_7 _ m_{15}	_	_	1	1

PI	Terms	0	1	3	7	8	9	11	15
$\overline{BC}$	0,1,8,9	x	x			x	x		
$\overline{B}D$	1,3,9,11		x	x			x	x	
CD	3,7,11,15			x	x			x	x

$$F = \overline{BC} + CD$$

Tabular Method with Don't Cares

	A	B	C	Y
$d0$	0	0	0	X
$m1$	0	0	1	0
$m2$	0	1	0	0
$m3$	0	1	1	0
$m4$	1	0	0	1
$m5$	1	0	1	1
$d6$	1	1	0	X
$m7$	1	1	1	1

Grouping

Logic Term	min term	binary representation	Numbers of '1's
$\overline{A} \cdot \overline{B} \cdot \overline{C}$	$d0$	000	0
$A \cdot \overline{B} \cdot \overline{C}$	$m4$	100	1
$A \cdot \overline{B} \cdot C$	$m5$	101	2
$A \cdot B \cdot \overline{C}$	$d6$	110	2
$A \cdot B \cdot C$	$m7$	111	3
$\overline{B} \cdot \overline{C}$	$(d0, m4)$	-00	0
$A \cdot \overline{B}$	$(m4, m5)$	10-	1
$A \cdot \overline{C}$	$(m4, d6)$	1-0	1
$A \cdot C$	$(m5, m7)$	1-1	2
$A \cdot B$	$(d6, m7)$	11-	2

Matching

Logic Term	min term	binary representation	Numbers of '1's
$\overline{B} \cdot \overline{C}^*$	$(d0, m4)^*$	-00	0
$A \cdot \overline{B}$	$(m4, m5)$	10-	1
$A \cdot \overline{C}$	$(m4, d6)$	1-0	1
$A \cdot C$	$(m5, m7)$	1-1	2
$A \cdot B$	$(d6, m7)$	11-	2
A	$(m4, m5, d6, m7)$	1–	1
A	$(m4, d6, m5, m7)$	1–	1

Essential Terms Table

Note: excludes the don't care terms

	$m4$	$m5$	$m7$
$\overline{B} \cdot \overline{C}$	X		
A	X	X	X