Symmetry in Integer Linear Programing

Naghmeh Shahverdi
University Of New Brunswick

Abstract

An integer linear program (ILP) is symmetric if it's variables can be permuted without changing the structure of the problem. Basic properties of linear representations of finite groups can be used to reduce symmetric linear programming to solving linear programs of lower dimension.

Introduction

Definition 1: Let $P \subset R^n$ be a polytope with integral vertices. P is **lattice-free** if and only if $P \cap R^n = vert\ P$ where vert P is the set of vertices of P.

Definition 2: Let $G \leq GL_n(Z)$ be a finite group and let G_z be the orbit of some point $z \in Z^n$. We call the convex hull of this orbit ($conv(G_z)$) an **orbit polytope**.

Definition 3: $z \in Z^n$ is called a **core point** if and only if it's orbit polytope is lattice –free i.e:

$$conv(G_z) \cap Z^n = G_z$$

We call the set of all core points of G its core set. ($core(G)$)

Theorem 1: Let $min_{x \in P \cap Z^n} f(x)$ be a convex integer optimization problem under integrality restrictions with a convex function f and a convex set $C \subset R^n$. Let G be a finite subgroup of the linear symmetry group of this instance. Then

$$min_{x \in C \cap Z^n} f(x) = min_{x \in C \cap Core(G)} f(x)$$

In other words, it suffices to search for a solution in the core set of the symmetry group.

Equivalent relation between core points(Fundamental core set)

Two points $x, y \in Z^n$ shall be called isomorphic if there exists a $g \in G$ such that $x - gy \in Fix_Z(G)$
Where

$$\boldsymbol{Fix(G) = \{x \in R^n : gx = x\ for\ all\ g \in G\}}$$

We dene a fundamental core set of G to be a set of equivalence class representatives of all core points.

How searching in the core set is possible?

Suppose $G \leq S_n$ is symmetry group of an LP. From representation theory, R^n can be decomposed into a direct sum of irreducible invariant subspaces.

$$\boldsymbol{R^n = V_1 \oplus \dots \oplus V_m}$$

Theorem 1: Let $G \leq GL_n(Z)$ be a finite group and let z be a core point. Then there always exist a constant $C(G)$ depending on the group and a G-invariant subspace V of R^n diferent from $Fix(G)$ such that

$$\|z|_V\| \leq C(G)$$

This theorem shows that, each core point have a small projection into at least one invariant subspace

Finite an infinite fundamental core set

Finite case:
It has been shown that for 2-homogeneous groups the fundamental core set is finite. A possible strategy to solve the optimization problem could consist of the following two steps:
1-We enumerate all layers whose affine hull intersects P by ascending objective value. These affine hulls are hyperplanes, for which intersection with P is much easier to determine than the intersection of P with the discrete layer.
2. We check for each layer L found in the first step whether L itself intersects P.

Infinite case:
We can find $C(G)$ in theorem 2 and make feasible region smaller by adding those new constraints.